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THE EFFECT OF A MEAN FLUID VELOCITY GRADIENT ON THE STREAMWISE VELOCITY VARIANCE OF A PARTICLE SUSPENDED IN A TURBULENT FLOW

L. M. LILJEGREN

Analytical Sciences Department, Battelle, Pacific Northwest Laboratories, Richland, WA 99352, U.S.A.

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Abstract—The effect of a mean fluid velocity gradient on the motion of a small solid particle suspended in a turbulent gas is analyzed using Fourier transform techniques. The presence of a mean fluid velocity gradient is shown to elevate the streamwise particle velocity variance above the level predicted without such gradients; the particle velocity variance in the direction normal to the flow is shown to be only indirectly affected by the existence of fluid velocity gradients. When the particle Stokes number is small, the streamwise particle velocity variance is elevated above the level predicted in flows without mean velocity gradients at a rate which is linearly proportional to both the particle Stokes number, α, and the ratio of the streamwise velocity gradient to the characteristic frequency of the energy-containing eddies in the turbulent field, G/ω_e. For particles with large Stokes numbers, the dominant mechanism by which particle streamwise velocity fluctuations are generated is through random interaction between the particle path and the mean fluid velocity gradient.

Key Words: particulate motion, turbulence, particle velocity variance, two-phase flow

1. INTRODUCTION

The motion of particles in uniform flow with no mean velocity gradients has been studied both experimentally (Wells & Stock 1983) and analytically (Tchen 1947; Lumley 1957; Csanady 1963; Chao 1964; Reeks 1977; Pismen & Nir 1978). All analyses predict that in the absence of velocity gradients:

1. The particle velocity variance cannot exceed the fluid velocity variance for any value of the particle Stokes numbers, α. The Stokes number is defined as the ratio of the frequency describing the energetic eddies in fluid turbulence as seen in a Lagrangian framework, ω_e, to the particle response frequency; i.e. $\alpha = \omega_e/\beta$.
2. The particle velocity variance decreases with increasing particle Stokes number.
3. The particle velocity variance will approach zero at large particle Stokes numbers, $\alpha \gg 1$.

The type of behavior described above has been observed in experiments conducted by Wells & Stock (1983). The turbulence in these experiments was generated by passing a uniform velocity field though a wire screen mesh, which resulted in a turbulence field with minimal mean velocity gradients.

Measurement of the particle velocity variance in naturally occurring flows suggests that the analyses cited provide incomplete descriptions of the random particle motions. It is frequently reported in the literature that the streamwise velocity variances of particles conveyed in gases exceed the velocity variances of the carrier fluid. Recently, Rogers & Eaton (1990) reported that the streamwise particle velocity variance for 90-μm glass beads conveyed in a vertical boundary layer exceeds the air velocity variance by approx. 20% throughout the boundary layer; the streamwise velocity variance of 50-μm particles was approximately equal to that of the air. In contrast, the transverse velocity variances for both size particles were significantly smaller than the variances measured for the air. Thus, while the magnitude of the transverse component of the particle velocity variance varied in the manner expected on the basis of the cited analyses, the streamwise component exhibits much larger variances. In fact, the streamwise component of the particle velocity variance exhibits two features which contradict the predictions of previous analyses:

1. The streamwise particle velocity variance exceeds the fluid velocity variance.
2. The streamwise particle velocity variance appears to increase with increasing particle Stokes number. That is, larger, less responsive particles exhibit higher velocity variances than smaller, more responsive particles.

Rogers & Eaton (1990) attempt to explain the discrepancy between observations and predictions based on Csanady's theory by assuming that the time and length scales of the turbulence as seen along the particle path are "stretched" relative to the Eulerian scales. This effectively decreases the particle Stokes number. However, both Csanady's original analysis and Rogers' and Eaton's modified analysis predict that the maximum possible particle velocity variance is equal to the fluid velocity variance and that the particle velocity variance decreases with increasing particle Stokes number. Consequently, the modification does not predict the two major qualitative differences between the measured and the predicted magnitude of the particle velocity variance.

Streamwise particle velocity variances which exceed streamwise fluid velocity fluctuations in pipe or duct flows have also been reported by Soo *et al.* (1960), Carlson & Peskin (1975), Tsuji & Morikawa (1982) and Steimke & Dukler (1983). Laser Doppler velocimetry measurements in the wake of a bluff body suggest that the intensity of streamwise particle velocity fluctuations increases with particle size in regions of high shear (Bachalo *et al.* 1987). Interestingly, in the cases in which the particle velocity variance in the direction transverse to the flow was also measured, the transverse velocity fluctuations did not exhibit comparable increases in magnitude. Instead, the transverse velocity fluctuations display the qualitative dependence on the particle Stokes numbers expected on the basis of published analyses describing the response of particles to turbulent fields.

The reports of surprisingly large particle velocity variances are sufficiently frequent to warrant a search for possible causes. In this paper, the motion of a small solid particle suspended in a gas undergoing mean shear will be examined. The analysis will rely on a number of assumptions pertaining to the particle dynamic equations, the mean velocity field, and the fluid turbulence spectrum as seen by the particle. Because of these assumptions, the results will be qualitative in nature. However, the analysis will show that the following behaviors may be possible when the carrier fluid exhibits a mean velocity gradient:

1. The particle velocity variance may exceed the fluid velocity variance.
2. The particle velocity variance may increase with increasing Stokes number.

The predictions of this analysis are consistent with the measurements reported by Rogers & Eaton (1990), Soo *et al.* (1960), Carlson & Peskin (1975), Steimke & Dukler (1983) and Tsuji & Morikawa (1982).

2. MOTION OF A SMALL PARTICLE SUSPENDED IN SHEAR FLOW

We shall consider the motion of a small solid spherical particle suspended in an incompressible, unbounded, homogeneous turbulent flow with mean fluid velocity in the x (for streamwise) coordinate direction which varies spatially with constant velocity gradient, G (i.e. $\bar{U}_x = Gy$, $\bar{U}_y = 0$ and $\bar{U}_z = 0$). The fluid turbulence will be assumed to be homogeneous and stationary, and the analysis will be restricted to the flows where the Reynolds number based on the particle diameter and the velocity difference between the particle and the fluid is $\ll 1$.

The motion of a small solid spherical particle in air may be described on the basis of a simplified equation of motion suggested by Lumley (1978):

$$\dot{V}_x + \beta V_x = F_x + \beta(\bar{U}_x + u_x)|_{\mathbf{x}_p} \tag{1}$$

and

$$\dot{V}_y + \beta V_y = F_y + \beta(\bar{U}_y + u_y)|_{\mathbf{x}_p}. \tag{2}$$

The quantity β is referred to as the particle cutoff frequency, which is the inverse of the particle response time τ_p; for a small sphere with diameter a, and particle Reynolds numbers based on the mean velocity difference between the fluid and the particles, $|(\bar{\mathbf{U}}-\bar{\mathbf{V}})|$, which is $\ll 1$, the particle response frequency β is given by the Stokes relation

$$\beta^{-1}=18\frac{\rho_f \nu}{\rho_p a^2}.$$

$\mathbf{F}$ is the body force exerted on the particle.

These equations are valid provided that the particle Reynolds numbers based on either the mean or fluctuating velocity difference between the particle and the fluid are small ($Re_p \ll 1$) and that the particle diameter, a, is much smaller than the Taylor microscale describing the fluid turbulence. That is, [1] and [2] govern the particle motion when

$$\text{for } Re_p=\frac{|(\bar{\mathbf{U}}-\bar{\mathbf{V}})\|a}{\nu}\ll 1,\quad Re_{p\Delta v}=\frac{\Delta v a}{\nu}\ll 1 \quad\text{and}\quad \frac{a^2}{\lambda^2}\ll 1,$$

where Δv is the r.m.s. difference between the fluid and particle fluctuating velocities and λ is the Taylor microscale.

Three forcing functions appear on the right-hand side of the particle dynamic equation. The body force, $\mathbf{F}$, is deterministic and has a direct effect on the mean particle velocity. Because its influence on the particle velocity is indirect, the body force will be neglected in the following analysis. The force exerted by the fluid turbulence, $\beta\mathbf{u}$, acts as a random excitation to the particle motion in both the streamwise, or x, and transverse, or y, directions. The third force is that exerted by the mean velocity gradient and is represented by the term $\beta\bar{\mathbf{U}}|_{x_p}$, where the mean velocity is evaluated at the current particle position. When the mean fluid velocity varies spatially, the particle samples the mean velocity field in a random fashion and the force exerted by the mean velocity on the particle becomes random. The chief goal of this analysis is to study the effect of the $\beta\bar{\mathbf{U}}$ term on the particle velocity variance.

It is possible to show that in the absence of body forces and when $\overline{U_y}=0$, that the mean transverse particle velocity is equal to zero, i.e. $\overline{V_y}=0$. As a result, the mean transverse particle location is also zero $\overline{Y_p}=0$. Consequently, the mean streamwise particle velocity is also identically zero for this case, i.e. $\overline{V_x}=0$.

Substituting the relation for the mean fluid velocity, $\overline{U_x}=Gy_p$, into the particle dynamic equations result in equations for the two components of the fluctuating particle velocity:

$$\dot{v}_x+\beta v_x=\beta(Gy_p+u_x)|_{x_p} \tag{3}$$

and

$$\dot{v}_y+\beta v_y=\beta u_y|_{x_p}. \tag{4}$$

Whenever the fluctuating fluid velocity vector $\mathbf{u}$ is random, the particle velocity vector $\mathbf{v}$ and position vector $\mathbf{x}_p$ will also be random. When the particle velocity vector as a function of time is known, the particle position vector may be found by integration:

$$\mathbf{x}_p(t)=\mathbf{x}_p(0)+\int_0^t \mathbf{v}(\tau)\,d\tau. \tag{5}$$

Evaluation of the expected value of both sides of [4] indicates that the expected value of the particle transverse velocity, v_y, equals zero at all times. This result may be used to evaluate [5] to show that the expected value of the transverse (or y) component of the particle position is constant and does not change with time. The particle position at time $t=0$ may be defined as zero without loss of generality, in which case the expected value of the particle position is also zero at all times.

3. TRANSVERSE PARTICLE VELOCITY

The full solutions for the particle dynamic equations [3] and [4] are, in general, time dependent, but the transverse particle momentum equations may be shown to be asymptotically stationary in the mean and variance provided that the turbulent flow field is homogeneous (Hinze 1962). Which is to say that at sufficiently long times, the solution for the transverse particle velocity has a constant mean value and a constant variance. The magnitude of the variance, $\overline{v_y v_y}$, may be found by use of a Fourier transform, as suggested by Chao (1964).

Let the Fourier transform and its inverse be defined as

$$\hat{V}_i(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} V_i(t)\,dt \quad \text{and} \quad V_i(t) = \int_{-\infty}^{\infty} e^{i\omega t}\hat{V}_i(\omega)\,d\omega. \qquad [6]$$

The transformed transverse particle velocity equation, [4], is

$$\hat{v}_y = \frac{\beta}{\beta + i\omega\,\hat{u}_y} \qquad [7]$$

and the spectrum of the transverse component of the particle velocity is

$$S_{v_y v_y}(\omega) = \frac{\beta^2}{\beta^2 + \omega^2} S_{u_y u_y}(\omega); \qquad [8]$$

where $S_{u_y u_y}$ is the fluid velocity spectrum in the transverse, or y, direction as sampled along the particle path and is defined as the Fourier transform of the fluid velocity autocorrelation $R_{u_y u_y}(\tau)$ as sampled along the particle path. By definition of the spectrum, the velocity variances may be obtained by integrating the spectrum:

$$\overline{u_y u_y} = \int_{-\infty}^{\infty} S_{u_y u_y}(\omega)\,d\omega \qquad [9]$$

and

$$\overline{v_y v_y} = \int_{-\infty}^{\infty} S_{v_y v_y}(\omega)\,d\omega = \int_{-\infty}^{\infty} \frac{1}{1 + \left(\dfrac{\omega}{\beta}\right)^2} S_{u_y u_y}(\omega)\,d\omega. \qquad [10]$$

The properties of this solution can be examined more easily by writing the integral in the form

$$\overline{v_y v_y} = \int_{-\infty}^{\infty} S_{v_y v_y}(\omega)\,d\omega = \int_{-\infty}^{\infty} \frac{1}{1 + \alpha^2\left(\dfrac{\omega}{\omega_e}\right)^2} S_{u_y u_y}(\omega)\,d\omega, \qquad [11]$$

where $\alpha = \omega_e/\beta$ and ω_e is the frequency of the energy-containing eddies following a fluid particle.

Equation [10] is identical to the relation obtained for all components of the particle velocity variance in the absence of mean velocity gradients by Tchen (1947) and Csanady (1963); the relation by Chao (1964) is similar but contains additional terms which are due to the added mass and Basset force terms which appear in the more complete particle momentum equation. Rigorous evaluation of [8] and [10] requires specification of the fluid velocity spectrum as sampled along the particle path. Because the particle path may only be described in a stochastic sense and is a function of both the random particle position and the characteristics of the fluid turbulence field, the problem is inherently nonlinear. The difficulties associated with rigorous evaluation of [10] are discussed by Lumley (1957) and Chao (1964).

Csanady (1963) proposed that the solution for the particle velocity [10] be evaluated by assuming that the fluid spectrum seen along the particle path is identical to the Eulerian spectrum, which was some characteristic frequency ω_E. This method did not account for the inherent nonlinearity of the problem, but provided qualitative predictions of particle velocity variances. Solutions that account for the inherent nonlinearity for the case where mean velocity gradients are absent are provided by Reeks (1977), Pismen & Nir (1978) and Nir & Pismen (1979).

Like the solution of Tchen (1947), Csanady (1963) and Chao (1964), this solution for the transverse particle velocity variance [11] does not contain any terms which include the mean velocity gradient, G. Thus, the existence of mean velocity gradients does not affect the particle velocity statistics in the transverse direction, except in an indirect manner by altering the shape of the fluid velocity spectrum or by affecting the path traveled by the particle. The difference between the solution presented and that which describes the particle transverse velocity variance in the absence of a mean velocity gradient is that the shape of the spectrum $S_{u_y u_y}(\omega)$ seen by the particle will differ depending on whether or not a mean velocity gradient exists. The difference in the apparent spectra will arise as a result of the appearance of Gy_p in the equation describing the particle streamwise velocity fluctuation, [3]; this term will affect the streamwise particle velocity and the particle position. However, a much more lengthy analysis than presented here is required to determine the exact manner in which the form of the spectra is affected by the differences in the path followed by a particle. Because the primary focus here is on the qualitative effect of a mean velocity gradient on the particle velocity variance, the nonlinearity involved in the evaluation of the particle velocity will be ignored.

It is possible to make some qualitative observations about the behavior of the particle transverse velocity variance on the basis of what is known about the particle velocity variance in the absence of mean velocity gradients. First, the particle velocity variance will approach the fluid velocity variance when the particle Stokes number α approaches zero. The behavior of the particle at the large Stokes number limit is more complicated. However, because the fluid velocity spectrum is strictly positive, the particle velocity variance in the transverse direction will approach zero at large Stokes numbers, provided that the frequency of the turbulence as seen along the particle path does not decrease or that it decreases relatively slowly with particle Stokes number. Reeks (1977) demonstrated that this is the case for homogeneous isotropic turbulence without mean velocity gradients, and that the particle velocity variance will decrease monotonically with particle Stokes number when there are no mean velocity gradients.

Further speculation as to the effect of the mean velocity gradient on the particle transverse velocity variance must await analysis of the effect of the mean velocity gradient on the particle streamwise velocity variance.

4. STREAMWISE PARTICLE VELOCITY

The particle displacement y_p which appears in the streamwise equation is nonstationary in variance; i.e. the variance in the particle position $\overline{y_p y_p}$ grows with time. This complicates analysis of the streamwise velocity variance. It is convenient to define a new variable w_x:

$$w_x = v_x - \bar{U}_x(y_p) = v_x = Gy_p. \quad [12]$$

The quantity w_x describes the difference between the velocity of a particle and the mean velocity of the fluid at the particle's current position and will be referred to as the particle relative velocity. The quantity w_x is of practical importance because it is the quantity measured in experimental studies using laser velocimetry to measure the particle velocity variance. This includes the experiments by Carlson & Peskin (1975), Steimke & Dukler (1983), Tsuji & Morikawa (1982) and Rogers & Eaton (1990).

Substituting $\mathrm{d}y_p/\mathrm{d}t = v_y$, into [12] and using [3] results in

$$\dot{w}_x + \beta w_x = -Gv_y + \beta u_x. \quad [13]$$

In the idealized shear layer under consideration, the mean velocity gradient does not vary spatially, and the mean fluid velocity fluctuation and particle velocities are equal to zero (i.e. $\bar{u}_x = \bar{v}_y = 0$). By taking the average of [13] and solving it, the stationary solution for the mean relative particle velocity w_x can be shown to be zero. Therefore, in this idealized problem the ensemble average of the particle velocity conditioned on its location is always equal to the mean fluid velocity evaluated at the current particle position. (In real flow situations, spatial variations in G or particle–wall interactions would affect both the mean particle velocity and the particle velocity fluctuations. Analysis of these factors is complicated and is not attempted here.)

Comparison of [13] and [4], the equations governing the transverse particle velocity fluctuations, reveals that the existence of mean velocity gradients must affect the particle velocity statistics in some manner. However, the order of magnitude of the influence of the mean velocity gradient on the particle velocity variance must be estimated in order to determine the conditions in which the presence of the mean velocity gradient has a detectable effect on the particle statistics.

The variance of the relative velocity can be shown to be stationary at long times. Consequently, the relative velocity spectrum may be obtained by applying Fourier transform techniques to solve the equation for the streamwise particle relative velocity. This results in

$$S_{w_x w_x}(\omega) = \frac{\beta^2}{\beta^2+\omega^2} S_{u_x u_x}(\omega) - \frac{\beta G}{\beta^2+\omega^2}[S_{u_x v_y}(\omega) + S^*_{u_x v_y}(\omega)] + \frac{G^2}{\beta^2+\omega^2} S_{v_y v_y}(\omega), \qquad [14]$$

where the asterisk denotes the complex conjugate.

The first term in the particle streamwise velocity spectrum [14] corresponds to the streamwise particle velocity spectrum in the absence of velocity gradients. The second and third terms represent the effects of velocity gradients on the particle motion. Two fluid–particle velocity cross-spectra describing the transverse particle velocity variance appear in the equation; these must be specified before the effect of shear on the particle motion may be described.

Substituting for the cospectrum $S_{u_x u_y}$ in terms of the fluid spectrum leads to

$$(S_{u_x v_y}(\omega) + S^*_{u_x v_y}(\omega)) = 2\,\mathscr{Re}(S_{u_x v_y}(\omega))$$
$$= \frac{2\beta^2}{\beta^2+\omega^2}\left[\mathscr{Re}(S_{u_x u_y}(\omega)) + \frac{\omega}{\beta}\,\mathscr{Im}(S_{u_x u_y}(\omega))\right]. \qquad [15]$$

Little is known about the quadrature spectral density function $\mathscr{Im}(S_{u_x u_y}(\omega))$. However, the imaginary portion of the Fourier transform of a real function must be zero whenever the function is symmetric in time; i.e. $\mathscr{Im}\hat{f}(\omega) = 0$ when $f(\tau) = f(-\tau)$. In the absence of evidence to suggest that the fluctuations in one direction lead those in the other, the quadrature spectral density will be assumed to be equal to zero. The contribution of this term can be evaluated at a later date if more is learned about the form of the quadrature spectral density in turbulent shear flows.

In addition, the integral of the fluid cross-power spectra is equal to the velocity covariance as seen along the particle path:

$$\int_{-\infty}^{\infty} S_{u_x u_y}(\omega)\,d\omega = \overline{u_x u_y}. \qquad [16]$$

Because the sign of the fluid velocity covariance is generally in the opposite direction to that of the mean shear, G, the total area under the envelope of the function $-G(S_{u_x u_y} + S^*_{u_x u_y})$ as seen along the particle path is expected to be positive. In addition, measurements of the fluid Eulerian cross-power spectrum indicate that the product $-G(S_{u_x u_y} + S^*_{u_x u_y})$ is positive at $\omega = 0$ and decays at high frequencies (Champagne *et al.* 1970). The exact shape of the fluid Lagrangian cross-power spectrum is unknown. But it is reasonable to expect that the magnitude of the Lagrangian cross-power spectrum seen along the particle trajectory would have a maximum at low frequencies and that it would decay in magnitude at high frequencies in a manner similar to the Eulerian turbulence spectrum.

The integral of the transverse particle spectrum $S_{v_y v_y}$, is related to the fluid transverse spectrum show in [8]. Consequently, because the fluid velocity spectrum is positive at all frequencies, the transverse particle velocity spectrum will also be strictly positive. This ensures that the contribution of the third term in [14] to the streamwise particle velocity variance is always positive.

Substituting the relation for the transverse velocity spectrum [8] results in:

$$\frac{G^2}{\beta^2+\omega^2} S_{v_y v_y}(\omega) = \frac{G^2\beta^2}{(\beta^2+\omega^2)(\beta^2+\omega^2)} S_{u_x u_y}(\omega). \qquad [17]$$

Substituting [15–17] into [14] allows the particle relative variance to be expressed in terms of the fluid spectra as

$$\overline{w_x w_x} = I_1 + I_2 + I_3, \qquad [18]$$

where

$$I_1 = 2\int_0^{\infty} \frac{1}{1+\left(\frac{\omega}{\beta}\right)^2} S_{u_x u_x}(\omega)\,d\omega \qquad [19]$$

$$I_2 = -2\frac{G}{\beta}\int_0^\infty \frac{1}{\left[1+\left(\frac{\omega}{\beta}\right)^2\right]^2} 2\,\mathscr{Re}[S_{u_x u_y}(\omega)]\,\mathrm{d}\omega \qquad [20]$$

and

$$I_3 = 2\frac{G^2}{\beta^2}\int_0^\infty \frac{1}{\left[1+\left(\frac{\omega}{\beta}\right)^2\right]^2} S_{u_y u_y}(\omega)\,\mathrm{d}\omega. \qquad [21]$$

The first terms, I_1, is equal to the particle velocity variance in a flow without velocity gradients. The next two terms correspond to the additional particle velocity variance due to the presence of fluid velocity gradient. The term I_2 describes the additional level of particle velocity fluctuations due to the existence of a fluid velocity covariance, $\overline{u_x u_y}$, which is often large in regions of high mean shear. The term I_3 describes the degree to which random fluid motions in the transverse flow direction lead to increased particle velocity fluctuations. These elevated particle velocity fluctuations result from the random nature of the particle position, which causes it to sample the mean fluid velocity field in a random manner.

The exact shapes of the spectra in [19]–[21] are not completely understood, both because the shapes of spectra in shear flows have not been characterized extensively and because the spectra as seen by the particle are distorted by the particle motion. However, general features of the turbulence spectrum in homogeneous isotropic turbulence will be described to provide a framework for evaluation of these integrals.

Tennekes & Lumley (1972) suggest that when the Reynolds number describing the fluid flow field is large and the fluid turbulence is isotropic, the Lagrangian velocity spectrum may be written in the form

$$S_{u_x u_x} = S_{u_y u_y} = u\mathscr{L}\frac{X_{ii}}{3}, \qquad [22]$$

where

$$X_{ii}(\omega) = \pi^{-1} \qquad \text{for } \left|\frac{\omega\mathscr{L}}{u}\right| < C_0,$$

$$X_{ii}(\omega) = \pi^{-1}\left(\frac{\omega_e}{\omega}\right)^2(\omega) = 0 \qquad \text{for } C_0 < \left|\frac{\omega\mathscr{L}}{u}\right| < \left|\frac{\omega_d\mathscr{L}}{u}\right|$$

and

$$X_{ii}(\omega) = 0 \qquad \text{for } \left|\frac{\omega_d\mathscr{L}}{u}\right| < \left|\frac{\omega\mathscr{L}}{u}\right|,$$

where $\mathscr{L}$ is the Lagrangian integral scale of the turbulence, the quantity ω_d is the viscous cutoff frequency and C_0 is a constant that can be determined by requiring that the integral of the spectrum reproduce the velocity variance.

Tennekes & Lumley (1972) suggest that the ratio of the viscous cutoff frequency to the frequency of the energetic eddies can be approximated using the expression

$$\frac{\omega_d}{\omega_e} = 0.31\sqrt{\mathrm{Re}_{\mathscr{L}}}. \qquad [23]$$

The characteristic frequency of the energetic eddies is related to the velocity and length scale as

$$\omega_e = \frac{C_0 u}{\mathscr{L}}, \qquad [24]$$

where u is the characteristic fluid velocity fluctuation.

When the turbulence is isotropic the following relation holds:

$$u^2 = \overline{u_x u_x} = \overline{u_y u_y}. \qquad [25]$$

The requirement that the integral of the spectrum reproduce the appropriate velocity variance, [9] and [10], imposes the following condition:

$$C_0 = \frac{3\pi}{2\left(2 - \dfrac{\omega_e}{\omega_d}\right)}. \qquad [26]$$

This spectrum has three major features: (1) it displays a maximum at zero frequency; (2) it decays monotonically with frequency; and (3) it has a viscous cutoff frequency above which the spectrum is zero. Measurements of some components of the turbulence spectrum tensor in pure shear were reported by Champagne *et al.* (1970). The turbulence spectra in shear flows differed from those measured in homogeneous turbulence.

Nevertheless, the turbulence in shear flows shares some qualitative features with homogeneous turbulence including the existence of a cutoff frequency, ω_d, and a characteristic frequency, ω_e. Thus, even in a shear flow, the magnitude of the integrals described above depends strongly on the characteristic frequency of the energy-containing eddies in the flow, ω_e, the particle cutoff frequency, β, and the total area under the velocity spectrum, which is equal to the fluid velocity variance, u^2.

It is convenient to reformulate the three terms I_1, I_2 and I_3 contributing to the particle velocity variance in dimensionless form using the particle Stokes number, $\alpha = \omega_e/\beta$, and the velocity variances and covariances. The three terms become:

$$I_1 = \overline{u_x u_x}\mathscr{I}_1(\alpha), \qquad [27]$$

$$I_2 = -2\left(\frac{G}{\omega_e}\right)\overline{u_x u_y}[\alpha\mathscr{I}_2(\alpha)] \qquad [28]$$

and

$$I_3 = -2\left(\frac{G}{\omega_e}\right)^2\overline{u_y u_y}\,[\alpha^2\mathscr{I}_3(\alpha)], \qquad [29]$$

where

$$\mathscr{I}_1(\alpha) = 2\overline{u_x u_x}^{-1}\int_0^\infty \frac{\omega_e}{1+\alpha^2\tilde{\omega}^2} S_{u_x u_y}(\tilde{\omega})\,d\tilde{\omega}, \qquad [30]$$

$$\mathscr{I}_2(\alpha) = 2\overline{u_x u_y}^{-1}\int_0^\infty \frac{\omega_e}{(1+\alpha^2\tilde{\omega}^2)^2}\mathscr{Re}(S_{u_x u_y}(\tilde{\omega}))\,d\tilde{\omega} \qquad [31]$$

and

$$\mathscr{I}_3(\alpha) = 2\overline{u_y u_y}^{-1}\int_0^\infty \frac{\omega_e}{(1+\alpha^2\tilde{\omega}^2)^2} S_{u_x u_y}(\tilde{\omega})\,d\tilde{\omega}. \qquad [32]$$

The frequency of the fluid velocity fluctuations has been made dimensionless using the characteristic frequency of the energy-containing eddies $\tilde{\omega} = \omega/\omega_e$. By definition of the power and cross-power spectra, [9], [10] and [16], the functions $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_3 \to 1$ in the limit $\alpha \to 0$.

Two parameters appear in [17]–[19]. The first, G/ω_e, describes the relative importance of the shear forcing term in the particle dynamic equation. The second is the particle Stokes number, α. The effect of the Stokes number on the particle velocity variance is somewhat complicated. However, it is possible to determine the order of magnitude of the effect of the presence of the mean velocity gradient on the streamwise velocity at the limit of low particle Stokes number ($\alpha \ll 1$).

At the low Stokes number limit, where $\alpha \to 0$, the streamwise fluid velocity spectrum as seen by the particle is often approximated using the Lagrangian fluid spectrum proposed by Tennekes & Lumley (1972), which was described above. Evaluation of the integral using this spectrum is provided in the appendix; the evaluation assumes that the ratio of the viscous cutoff frequency to the frequency of the energetic eddies ω_d/ω_e is infinite. In this case, the particle streamwise velocity variance in the absence of shear is $\overline{w_x w_x} = I_1 = \overline{u_x u_x}[1 - \alpha\pi/4 + O(\alpha^2)]$. The coefficient for the second-order term [i.e. the $O(\alpha)$ term] depends strongly on the postulated form of the spectrum as seen by the particle. The numerical value of $\pi/4$ results from assuming that the viscous cutoff frequency ω_d is infinite. Nonetheless, in the absence of shear, the particle velocity variance is expected to decrease with increasing small values of the Stokes number.

The spectrum for the fluid covariance is expected to be bounded and to decrease sufficiently rapidly at large frequencies to cause the integral to be bounded. If so, it can be shown that $I_2 = -2\alpha(G/\omega_e)\overline{u_x u_y}$ to first order in α; this quantity is positive in magnitude because the shear rate G and the Reynolds stresses $\overline{u_x u_y}$ are always of opposite sign. Consequently, the Reynolds stresses cause the streamwise particle velocity variance to increase above the level expected in the absence of velocity gradients. The transverse fluctuations enhance the particle velocity variance as $I_3 = \alpha^2(G/\omega_e)^2\overline{u_y u_y}$, which is a higher-order effect.

The leading order terms describing the particle velocity variance become

$$\frac{\overline{w_x w_x}}{\overline{u_x u_x}} = 1 + \alpha\left(2\frac{|G\overline{u_x u_y}|}{\omega_e \overline{u_x u_x}} - \frac{\pi}{4}\right) + O(\alpha^2). \qquad [33]$$

Here the absolute value of the product of the shear rate and the Reynolds stress is shown in order to emphasize the tendency of this term to enhance the particle velocity variance.

In contrast, the transverse velocity fluctuations can be approximated to leading order by

$$\frac{\overline{v_y v_y}}{\overline{u_y u_y}} = 1 - \alpha\frac{\pi}{4} + O(\alpha^2). \qquad [34]$$

Examination of [33] indicates that at sufficiently large levels of shear, the streamwise particle velocity fluctuations may increase with α and many exceed the fluid velocity fluctuations. This is qualitatively similar to the behavior observed by Rogers & Eaton (1990), and is a behavior which has not been predicted by previous analyses of particle response to fluid turbulence. In contrast, when there is no mean velocity gradient the particle streamwise velocity variance decreases with increasing particle Stokes number. It is interesting to note that the streamwise velocity variance may increase with Stokes number when there is a velocity gradient, while the transverse velocity variance decreases with Stokes number. This is also qualitatively similar to results reported by Rogers & Eaton (1990) and others.

Analysis in the large particle Stokes number limit is more complicated and full treatment is not possible using the linear techniques applied here. However, it is possible to compare the order of magnitude of each of the three terms contributing to the particle velocity variance in this limit. This analysis will suggest the conditions in which a more complete analysis may be warranted.

Particles with large Stokes numbers respond only to the lowest frequency fluctuations of the spectrum as seen along the particle path. Because of this, it is reasonable to assume that the spectrum is constant:

$$S(\tilde{\omega}) \approx S(0). \qquad [35]$$

Despite the gross oversimplification involved in [35] the approximation is sufficiently detailed for the purpose of estimating the relative magnitude of the three terms contributing to the velocity variance. The spectra selected share some important features with actual spectra. First, power spectra of real functions, such as the fluid velocity variances, are by definition symmetric and the real part of the cross-power spectra of a real function is also symmetric. Consequently, the first derivative of both the spectra and cospectra is zero at the origin; i.e. $\mathrm{d}S_{u_i u_j}(\omega)/\mathrm{d}\omega = 0$ at $\omega = 0$. In addition, it may be assumed that the fluid velocity spectra and cospectra are bounded and decay at large ω. Consequently, the contribution of the higher frequency components to the particle velocity variance will be small relative to the contribution of the components near $\omega = 0$. The mathematical requirement that the integral reproduce the variance may be relaxed in this case without affecting the result because only a small portion of the spectrum influences the final results.

The definition of the Fourier transform requires that the magnitude of the spectrum at $\omega = 0$ be

$$S_{u_i u_j}(0) = \frac{\overline{u_i u_j}\tau_{\mathscr{L}ij}}{\pi}, \qquad [36]$$

where $\tau_{\mathscr{L}ij}$ represents an integral time scale as seen along the particle path.

The magnitude of this time scale can be shown to depend on the path followed by the particle and can be determined only by performing a full analysis that accounts for the inherent nonlinear nature of the problem posed. Reeks (1977) and Pismen & Nir (1978) have shown that when the turbulence is isotropic and there is no mean velocity gradient, the integral time scale as seen along the particle path will increase with increasing particle Stokes number. However because mean velocity gradients tend to elevate the particle velocity variance this trend in the integral time scale may be offset or even reversed when a mean velocity gradient is present.

Despite the ambiguity in the actual magnitude of the time scale as seen by the particle, it is possible to estimate the relative magnitude of the contributions of the terms I_1, I_2 and I_3 to the velocity variance. It is sufficient to know that a time scale exists. Assuming that the velocity spectrum may be described using [35], the first-order contribution to the I_1 term is

$$I_1 = \alpha^{-1}\pi S_{u_x u_x}(0)\omega_e = \alpha^{-1}\overline{u_x u_x}\tau_{\mathscr{L}x}\omega_e, \qquad [37]$$

where $\tau_{\mathscr{L}x}$ is the longitudinal integral time scale for the fluid turbulence as seen along the particle path and $S_{u_x u_x}(0)$ is the value of the fluid spectrum at the origin.

This term is $O(\alpha^{-1})$; therefore, its magnitude tends to decrease with increasing particle Stokes number. Variation of the timescale of the turbulence as seen by a particle with higher Stokes number would modify the rate at which the magnitude of I_1 actually decreases with Stokes number.

The first-order contribution of the Reynolds stresses integral (I_2) to the particle velocity variance can be shown to be

$$I_2 = u^2\left[\frac{G}{\omega_e}\frac{\overline{u_x u_y}}{u^2}\tau_{\mathscr{L}xy}\omega_e + O(\alpha^{-1})\right]. \qquad [38]$$

This terms is $O(1)$, and provides a larger contribution to the particle velocity variance than the I_1 term. This is true, irrespective of the nonlinear variation of the integral time scale of the turbulence as seen along the particle path.

Finally, the first-order contribution due to the transverse fluid fluctuations is

$$I_3 = u^2\left[\alpha\left(\frac{G}{\omega_e}\right)^2\left(\frac{\overline{u_y u_y}}{u^2}\right)\frac{\tau_{\mathscr{L}y}\omega_e}{2} + O(1)\right]. \qquad [39]$$

The form of this term suggests that the particle velocity fluctuations will tend to increase linearly with particle Stokes number, and it is likely that the particle velocity variance could be larger than the fluid velocity variance when the particle Stokes number is sufficiently large. However, it is unlikely that the particle velocity variance will become unbounded. Examination of full nonlinear treatments of the particle motion such as those by Reeks (1977) indicate that a large mean particle velocity variance will tend to decrease the integral time scale of the turbulence spectrum as seen along the particle path. Consequently, it is not possible to determine whether the particle velocity variance increases, decreases, or approaches a constant with increasing Stokes number in a particular flow without performing a full nonlinear analysis. Such an analysis may be warranted when better information becomes available regarding the form of all the components of the velocity spectra in the presence of a mean shear layer.

However, it is possible to conclude that the streamwise velocity variance of the particle will tend to be much larger than the transverse velocity variance, because the transverse fluctuations will be on the order of the I_1 term contributing to the streamwise velocity variance which is $O(\alpha^{-1})$. By examining the relative orders of magnitude of the terms describing the three separate contributions to the particle velocity variance it is also possible to show that when the particle Stokes number is large, the influence of the fluid velocity gradient on the particle behavior can be extremely important.

The relative orders of magnitude of the three terms contributing to the particle streamwise velocity variance are

$$\frac{I_3}{I_1} = O\left\{\alpha^2\left(\frac{G}{\omega_e}\right)^2\frac{\overline{u_y u_y}}{\overline{u_x u_x}}\frac{\tau_{\mathscr{L}y}}{\tau_{\mathscr{L}x}}\right\} = O\left\{\left(\frac{G}{\beta}\right)^2\frac{\overline{u_y u_y}}{\overline{u_x u_x}}\frac{\tau_{\mathscr{L}y}}{\tau_{\mathscr{L}x}}\right\} \qquad [40]$$

and

$$\frac{I_2}{I_1} = O\left\{\alpha\left(\frac{G}{\omega_e}\right)\frac{\overline{u_x u_y}}{\overline{u_x u_x}}\frac{\tau_{\mathscr{L}xy}}{\tau_{\mathscr{L}x}}\right\} = O\left\{\frac{G}{\beta}\frac{\overline{u_x u_y}}{\overline{u_x u_x}}\frac{\tau_{\mathscr{L}xy}}{\tau_{\mathscr{L}x}}\right\}. \qquad [41]$$

It is likely that the integral time scales describing the streamwise and transverse velocity correlations, and the integral time scale describing the Reynolds stresses as seen along the particle path are all of comparable magnitude. Likewise the magnitude of the fluid velocity variances and the Reynolds stresses are of comparable order. Consequently, the relative orders of magnitude of the terms become

$$\frac{I_3}{I_1} = O\left\{\alpha^2\left(\frac{G}{\omega_e}\right)^2\right\} \qquad [42]$$

and

$$\frac{I_2}{I_1} = O\left\{\alpha\left(\frac{G}{\omega_e}\right)\right\}. \qquad [43]$$

These relations indicate that when the product $\alpha(G/\omega_e) > 1$, the influence of the mean fluid velocity gradient is more important in determining the streamwise particle velocity variance than the direct effect of the turbulent fluid velocity fluctuations in the streamwise direction. In very dilute particulate flows, the particles cannot affect the fluid flow field and the ratio (G/ω_e) is not affected by the presence of the particles. Consequently, there will be some critical magnitude of the particle Stokes number above which the presence of the mean velocity gradient cannot be neglected during analysis of the particle motions. Above this particle Stokes number, analyses in which the effect of the mean velocity gradient is neglected will provide predictions of the particle velocity variance which are qualitatively incorrect.

The new type of particle behavior described is important for two reasons. First, it suggests a physical mechanism which can explain numerous experimental observations in which the streamwise particle velocity variance is observed to exceed the transverse fluctuations. These include those by Rogers & Eaton (1990), where observed streamwise particle velocity fluctuations exceeded the fluid velocity fluctuations.

The behavior described in this analysis also directly affects the understanding of turbulence development in particulate flows. Analyses, such as those by Elghobashi & Abou-Arab (1983), generally indicate that the interaction of the fluid turbulence with the particles leads inevitably to an additional dissipation term in the fluid kinetic energy equation. However, an extension of the analysis of the particle motion can be performed to show that the variance of slip between the particle velocity and the fluid velocity (i.e. Δw_x) can exceed the fluid velocity variance. It can also be shown that the particle interaction term appearing in the kinetic energy equation proposed by Elghobashi & Abou-Arab (1983) can lead to the creation of additional fluid turbulent kinetic energy.

A physical explanation of the phenomena predicted here can be provided by considering the action of an "average" eddy on the motion of particles in the flow. When the rate of shear is positive, the eddies with negative transverse velocity tend to have positive fluctuating streamwise velocity, hence the negative fluid Reynolds stresses. Likewise, a particle at a location $y_p > 0$ would, on average, have a larger positive velocity relative to a particle located at $y_p = 0$. Thus, when caught in a strong downward eddy and carried to $y_p = 0$, a particle arriving from above, will, on average, have excess streamwise velocities relative to the local mean particle velocity. For a particle arriving from below, the scenario is reversed, leading to negative velocities. As a consequence, the total velocity variance, averaged over particles arriving from above and below, is larger than that measured when there is no fluid shear.

The effect described is more pronounced for particles with large Stokes numbers. Particles with infinitesimal Stokes numbers will adjust to the velocity of the eddy very rapidly and thus arrive at y_p with the velocity statistics characteristic of the fluid. Those with large Stokes numbers do not adjust quickly to the velocity of the fluid and will retain their excess velocity for finite lengths of time. At sufficiently large Stokes numbers, the particles may retain a velocity difference long after the eddy itself has decayed in strength, leading to enhanced particle velocity variances.

5. CONCLUSIONS

From the above discussion, two conclusions may be drawn regarding the streamwise velocity variance:

(i) The magnitude of the streamwise velocity variance of solid particles suspended in air is elevated by the presence of mean fluid velocity gradients, while the magnitude of the transverse fluctuations is relatively unaffected. The parameters governing the degree to which the particle streamwise velocity variance is enhanced are: (1) the particle Stokes number, $\alpha = \omega_e/\beta$, which is the ratio of the characteristic frequency of the energy-containing eddies to the particle cutoff frequency; and (2) an enhancement parameter G/ω_e which describes the influence of mean velocity gradients relative to that of the fluid turbulence fluctuations on the particle motion. The magnitude of the streamwise particle velocity variance increases proportionally with increases in either of these two parameters.

(ii) At large Stokes numbers, the random interaction of a suspended particle with the mean fluid velocity gradients dominates the motion of the particle and may lead to a streamwise particle velocity variance which exceeds that predicted in the absence of shear. In general, the difference between the velocity variance predicted in the absence of mean velocity gradients and that which occurs in the presence of mean velocity gradients can be very large. More importantly, the results of the analyses indicate that when the particle Stokes number is large, analyses that ignore the presence of the mean velocity gradients will provide qualitatively incorrect predictions of the particle velocity variance. This implies that analyses that account for the full nonlinear behavior of the particle dynamic equation should also include the existence of the mean velocity gradient, particularly when the particle Stokes number is large.

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APPENDIX

Evaluation of the Integrals $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_3$ for Small Stokes Number

Determination of the particle velocity variance to the turbulence field requires evaluation of the integrals appearing in [30–32]. Asymptotic expressions for the response of a particle with a small Stokes number are developed here. The turbulence spectrum is assumed to be of the form suggested by Tennekes & Lumley (1972), as described in [23–25]; the limits form as the flow Reynolds number approaches infinity is used in this analysis. That is, it is assumed that $C_0 = 3\pi/4$ and $\omega_e/\omega_d = 0$.

The first integral to be evaluated is $\mathscr{I}_1$ which provides the streamwise velocity fluctuation in the absence of mean velocity variations. A similar integral appears in [10], which describes the transverse velocity variance. Substitution of the spectrum to be evaluated into the first integral [30] yields

$$\mathscr{I}_1(\alpha) = 2\overline{u_x u_x}^{-1}\int_0^\infty \frac{\omega_e}{1+\alpha^2\tilde{\omega}^2} S_{u_x u_x}(\tilde{\omega})\,d\tilde{\omega}$$
$$= \frac{2C_0}{3\pi}\left[\int_0^1 \frac{1}{1+\alpha^2\tilde{\omega}^2}\,d\tilde{\omega} + \int_1^{\omega_d/\omega_e} \frac{1}{(1+\alpha^2\tilde{\omega}^2)\tilde{\omega}^2}\,d\tilde{\omega}\right]. \quad [A.1]$$

This may be integrated using the following identities:

$$\int \frac{dx}{1+\alpha^2 x^2} = \frac{\tan^{-1}(\alpha x)}{\alpha} \quad [A.2]$$

and

$$\int \frac{dx}{x^2(1+\alpha^2 x^2)} = \frac{-1}{x} - \alpha\tan^{-1}(\alpha x). \quad [A.3]$$

Resulting in

$$\mathscr{I}_1(\alpha) = \frac{2C_0}{3\pi}\left[\frac{\tan^{-1}(\alpha)}{\alpha} + \left(1-\frac{\omega_e}{\omega_d}\right) + \alpha\left(\tan^{-1}(\alpha) - \tan^{-1}\left(\frac{\alpha\omega_d}{\omega_e}\right)\right)\right]. \quad [A.4]$$

An approximation that is valid for small particle Stokes numbers and infinite flow Reynolds numbers can be obtained by substituting the Taylor series expansion for the tangent of a small quantity:

$$\tan^{-1}(\alpha) = \alpha - \frac{\alpha^3}{3} + \cdots \quad [A.5]$$

and by recognizing that for large values of α the limiting value of the inverse tangent is $\pi/2$.

These substitutions allow approximation evaluation of the expression for $\mathscr{I}_1$ using:

$$\mathscr{I}_1(\alpha)=\left(1-\frac{\pi}{4}\alpha+\frac{\alpha^2}{2}\right)+O(\alpha^3). \qquad [A.6]$$

This approximation is valid if the flow Reynolds number is sufficiently large to cause the quantity $\alpha\omega_d/\omega_e$ to approach infinity and if the particle Stokes number is $\ll 1$. The first two terms of this asymptotic expression are used in the main text to describe the qualitative effect of the presence of mean velocity gradients on the particle velocity fluctuations.

The contribution of the Reynolds stress to the streamwise velocity variance was shown to be represented by a second integral in the form:

$$\mathscr{I}_2(\alpha)=2\overline{u_x u_y}^{-1}\int_0^\infty \frac{\omega_e}{(1+\alpha^2\tilde{\omega}^2)^2}\mathscr{Re}(S_{u_x u_y}(\tilde{\omega}))\,d\tilde{\omega}. \qquad [A.7]$$

If it is assumed that the cross-spectrum can be described functionally using the form suggested by Tennekes & Lumley (1972) for the turbulence spectrum, which is modified by replacing the covariance $u_x u_y$ with the velocity variance, u^2, then this integral may be written as

$$\mathscr{I}_2(\alpha)=\frac{2C_0}{3\pi}\left[\int_0^1 \frac{\omega_e}{(1+\alpha^2+\tilde{\omega}^2)^2}d\tilde{\omega}+\int_1^{\omega_d/\omega_e}\frac{\omega_e}{(1+\alpha^2\tilde{\omega}^2)^2\tilde{\omega}^2}d\tilde{\omega}\right]. \qquad [A.8]$$

The integrals may be evaluated by recognizing the following two standard integrals:

$$\int\frac{dx}{x^2(1+\alpha^2x^2)^2}=-\left[\frac{-1}{x}+\alpha\tan^{-1}(\alpha x)\right]-\alpha^2\left[\frac{x}{2(1+\alpha^2x^2)}+\frac{1}{2\alpha}\tan^{-1}(\alpha x)\right] \qquad [A.9]$$

and

$$\int\frac{dx}{(1+\alpha^2x^2)^2}=\frac{x}{2(1+\alpha^2x^2)}+\frac{1}{2\alpha}\tan^{-1}(\alpha x), \qquad [A.10]$$

so

$$\mathscr{I}_2(\alpha)=\frac{2C_0}{3\pi}\frac{1}{2}\left\{\frac{1}{1+\alpha^2}+\frac{\tan^{-1}(\alpha)}{\alpha}+2\left(1-\frac{\omega_e}{\omega_d}\right)+3\alpha\left[\tan^{-1}(\alpha)-\tan\left(\frac{\alpha\omega_d}{\omega_e}\right)\right]+\alpha^2\left[\frac{1}{1+\alpha^2}-\frac{\left(\frac{\omega_e}{\omega_d}\right)^2}{\left(\frac{\omega_e}{\omega_d}\right)^2+\alpha^2}\right]\right\}. \qquad [A.11]$$

This may be approximated using the expression

$$\mathscr{I}_2(\alpha)=1+\frac{3\pi}{2}\alpha+O(\alpha^2). \qquad [A.12]$$

As for the previous approximation of $\mathscr{I}_1$, this relation to $\mathscr{I}_2$ holds if the flow Reynolds number is sufficiently large that the quantity $\alpha\omega_d/\omega_e\to\infty$ and if the particle Stokes number $\alpha\ll 1$.

The first term in this relation is used in the discussion of the behavior of particles with small Stokes numbers. While a spectrum shape has been assumed here, it can be shown that the first term in the equation is unaffected by the form of the cross-spectrum, while the $O(\alpha)$ and higher terms will be strongly affected by the shape of the spectrum.

The third integral does not contribute to the low particle Stokes number solution, but is evaluated here for completeness:

$$\mathscr{I}_3(\alpha)=2\overline{u_y u_y}^{-1}\int_0^\infty\frac{\omega_e}{(1+\alpha^2\tilde{\omega}^2)^2}S_{u_y u_y}(\tilde{\omega})\,d\tilde{\omega}. \qquad [A.13]$$

This integral can be evaluated in a manner identical to $\mathscr{I}_2$, and can be represented using the relation

$$\mathscr{I}_3(\alpha)=1+\frac{3\pi}{2}\alpha+O(\alpha^2). \qquad [A.14]$$